STRUCTURE DE GRAPHES : MINEURS ET ARBRES INDUITS

Claire HILAIRE

Encadrants : Marthe BONAMY et Cyril GAVOILLE

4 Juillet 2023

Structure of Graphs: Minors and Induced Trees

Claire Hilaire

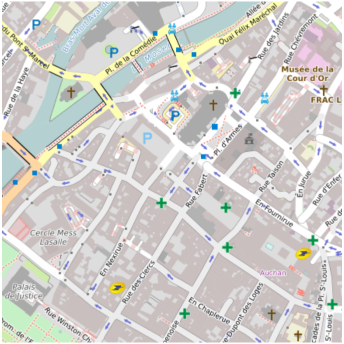
Advisors: Marthe Bonamy and Cyril Gavoille

July 4th, 2023

Université de Bordeaux

Laboratoire Bordelais de Recherche en Informatique LaBRI

Graphs in real life



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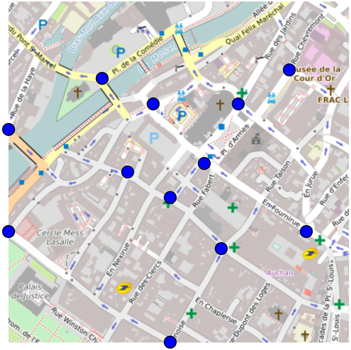

With a given budget, can we build bike lanes such that a path between two points of interest using those lanes is at most $2\times$ longer than using all the roads?

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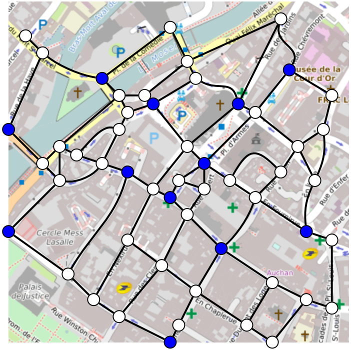

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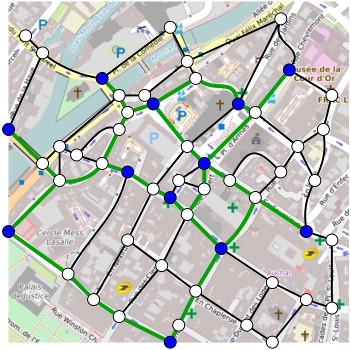

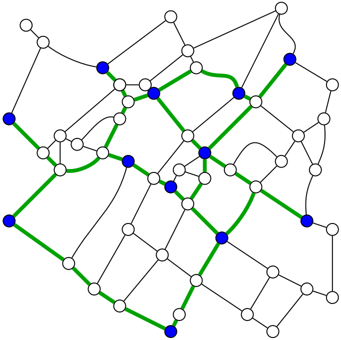

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Structure of graphs

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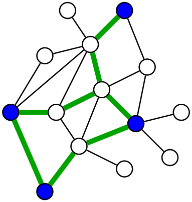

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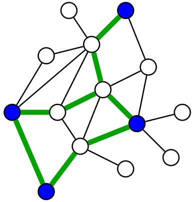

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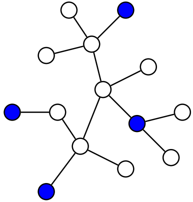

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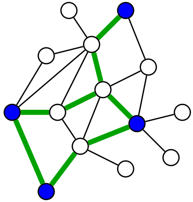

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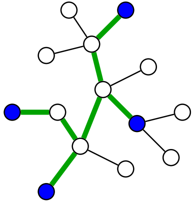
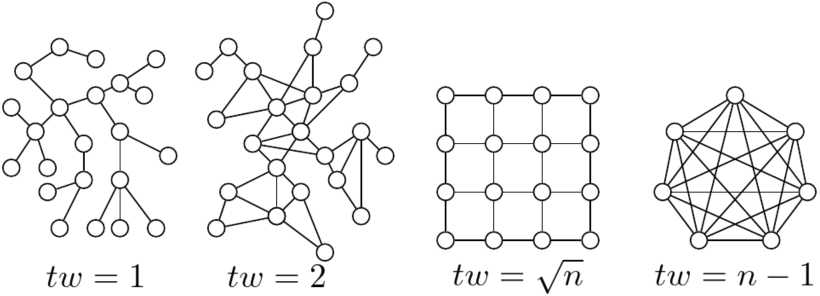


Easy!

The treewidth

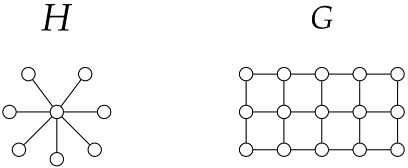
The treewidth of a graph G ($tw(G)$) measures how far from a tree G is.



Bounded treewidth generalizes trees.

Graph minors

H is a minor of G if H can be obtained from G by
- taking a subgraph
- contracting edges



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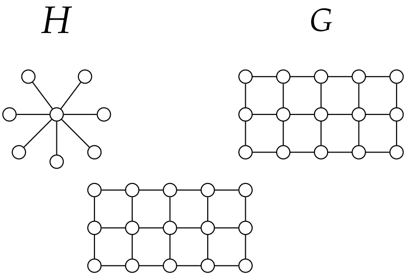

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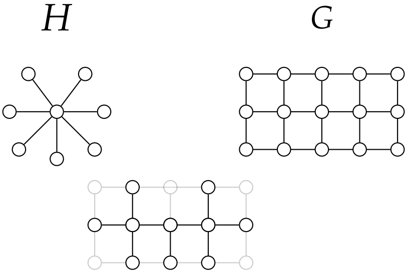

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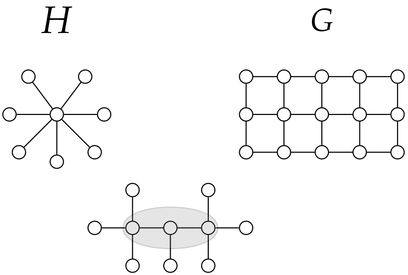

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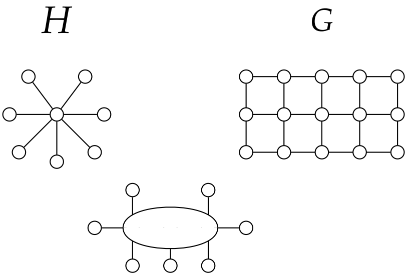

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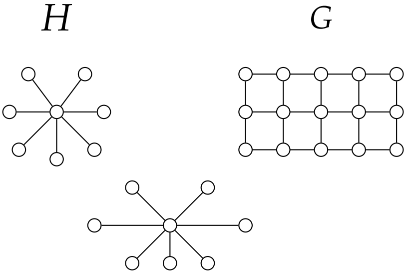

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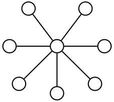
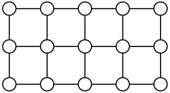


G



Robertson and Seymour (1986)

- G has no fixed planar H as minor $\Rightarrow$ G has bounded treewidth.

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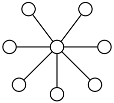

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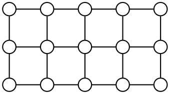

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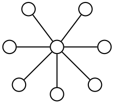

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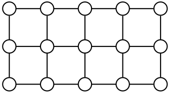

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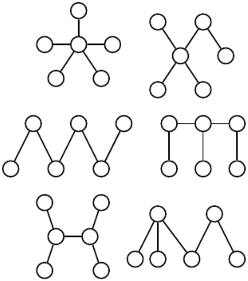
Every planar n-graph is minor of a $2n \times 2n$-grid.

Minor-universal graph

Let $\mathcal{F}$ be a family of finite graphs.



6-trees

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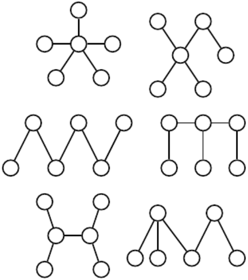

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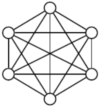

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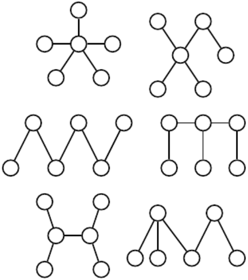

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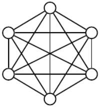

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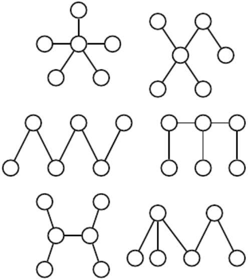

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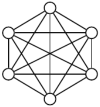
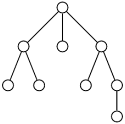
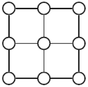


none



universal tree



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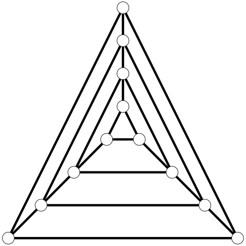
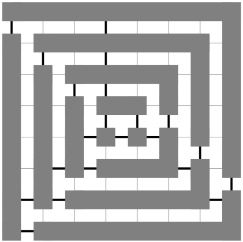


nested triangles



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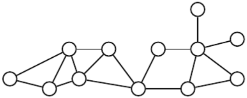

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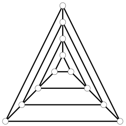

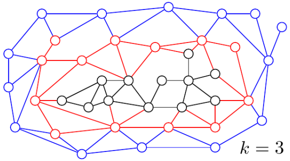

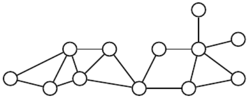

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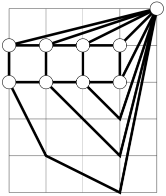
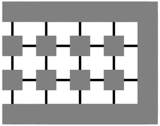


grid drawing

grid minor



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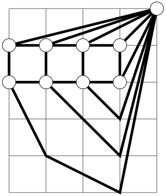
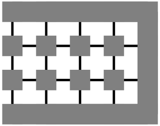


grid drawing $\xrightarrow{\times\sqrt{n}}$ grid minor



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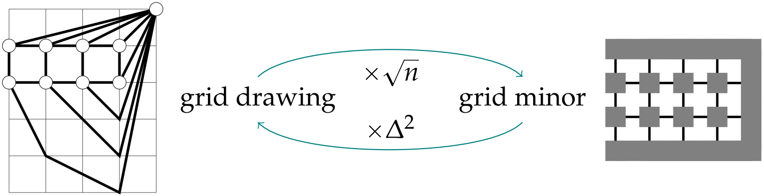
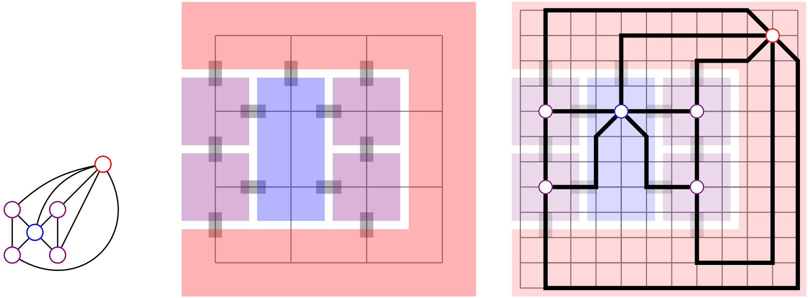
- minor of n-area-grid $\Rightarrow$ drawing on the $\Delta^2 n$-area-grid. [GH23+]



From universal grid to grid drawing

Gavoille and H. 2023+

Minor of n-area-grid and max degree Δ
$\Rightarrow$ drawing on the $\Delta^2 n$-area-grid.



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Graphs on surfaces

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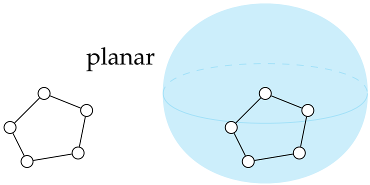

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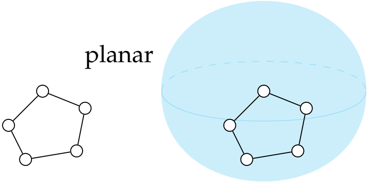

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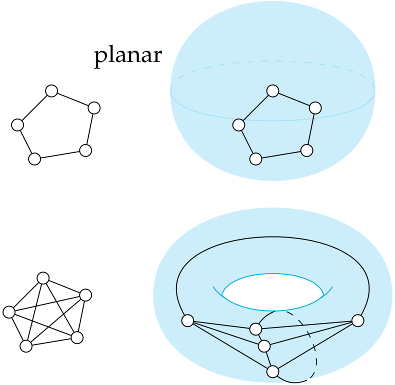

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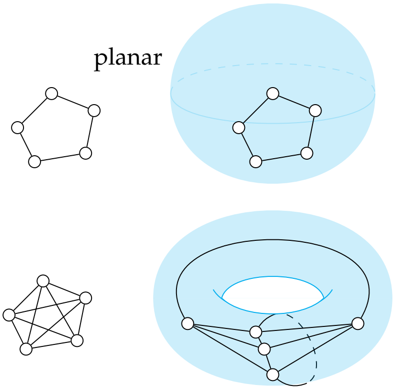
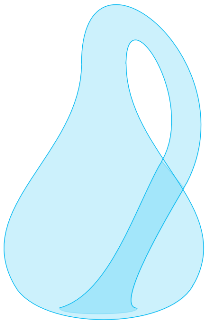




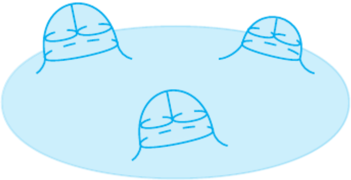
Classification of surfaces

Every connected surface without boundary is homeomorphic to either:

- An **orientable** surface of Euler genus $2g \geqslant 0$:

here $g = 3$

- A **non-orientable** surface of Euler genus $g \geqslant 0$:



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Gavoille and H. (2023+)

For every n and every surface Σ of Euler genus $g \geqslant 1$, there is a graph embedded on Σ with $O(g^2(n+g)^2)$ vertices minor-universal for the n-graphs embeddable on Σ.

Minor-universal grid for planar graphs

Robertson, Seymour and Thomas (1994)

For every n, there is a planar graph on $O(n^2)$ vertices minor-universal for the planar n-graphs.

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Step 2: Getting the grid

Every Hamiltonian planar N-graph is a minor of the $N \times N$-grid.

Construction of a Hamiltonian major

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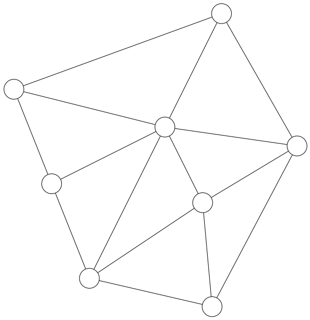

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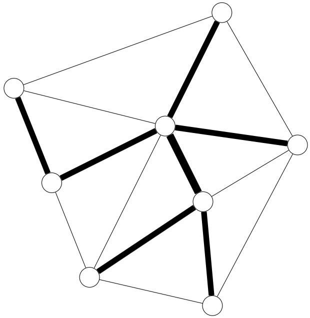
Every planar n-graph is a minor of a Hamiltonian planar $2n$-graph.



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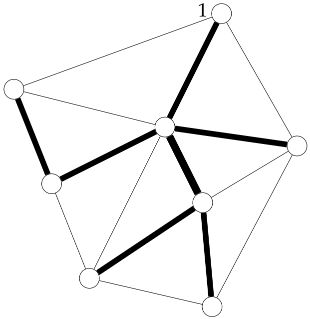

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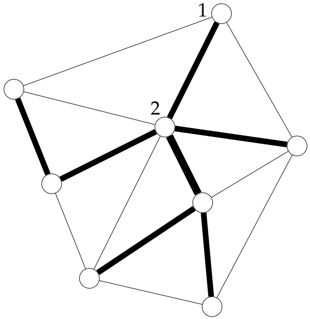

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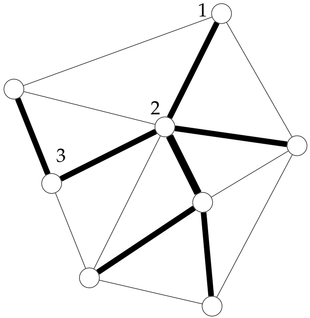

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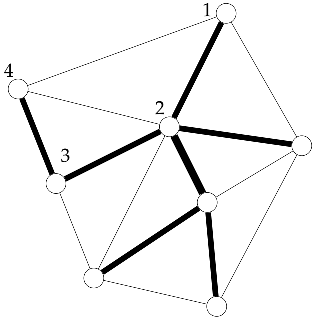

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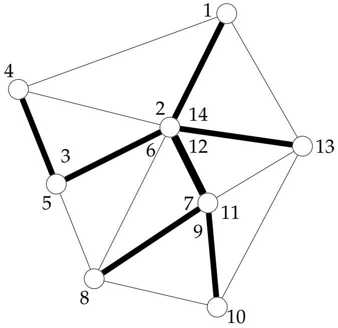

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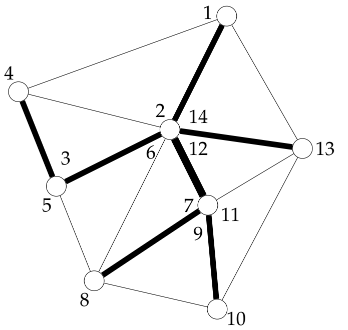

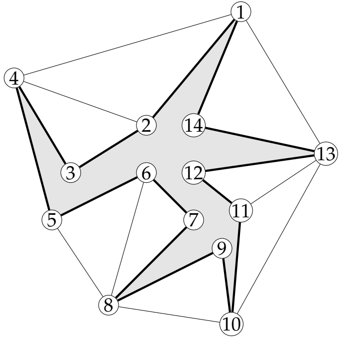

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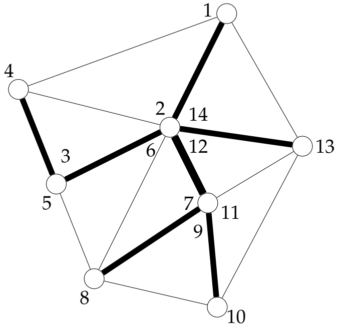

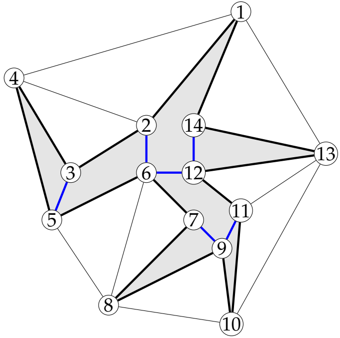

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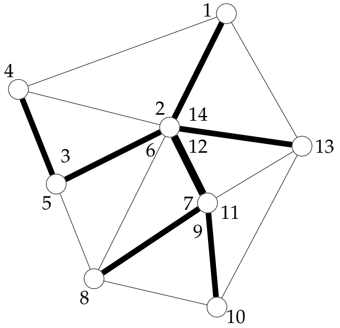

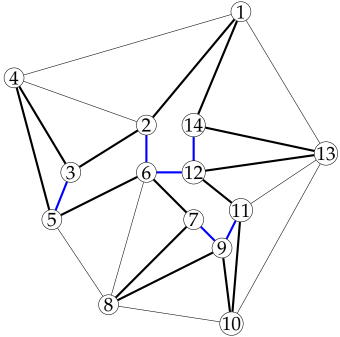

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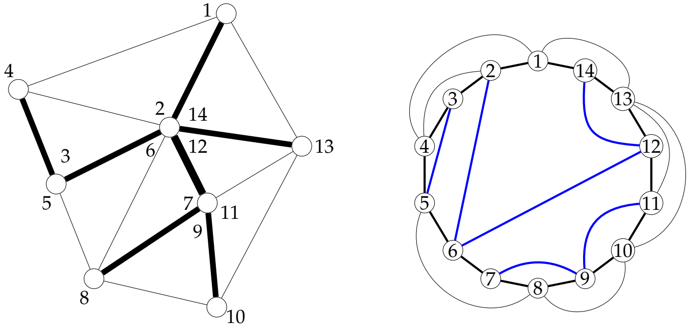
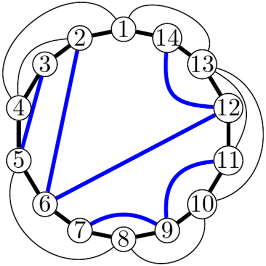
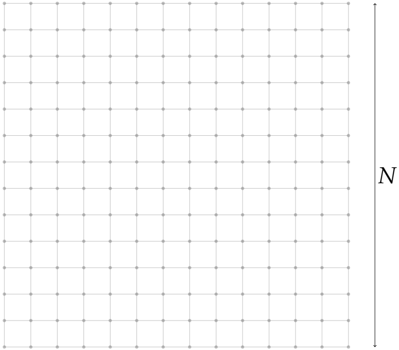


Construction of a grid major

Step 2: Getting the grid

Every Hamiltonian planar N-graph is a minor of the $N \times N$-grid.





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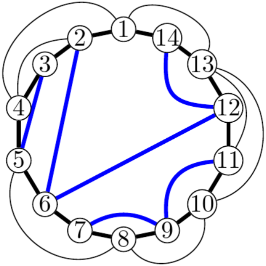

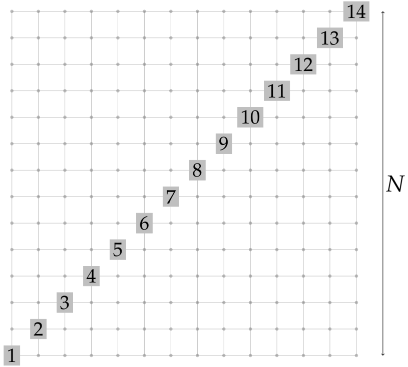

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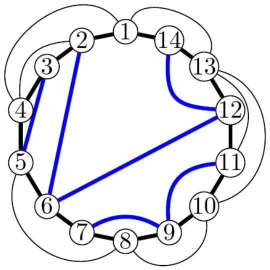

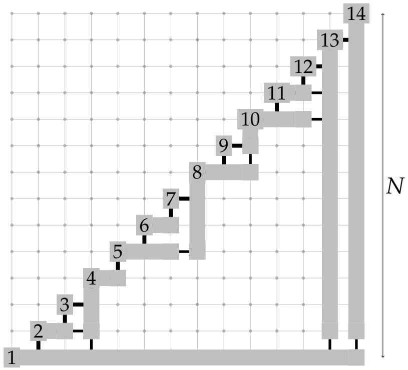

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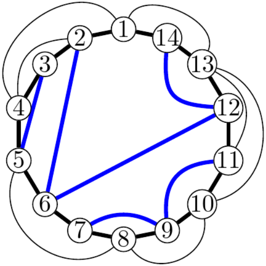
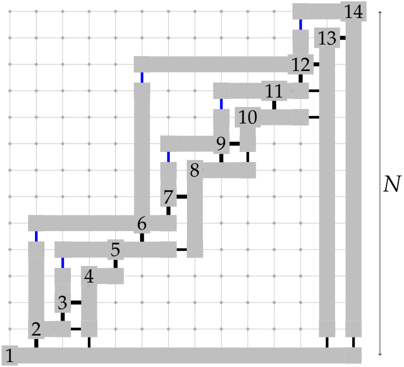
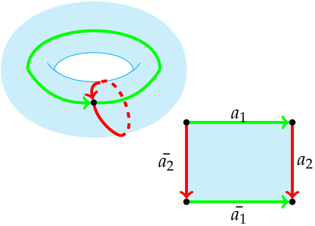




Polygonal schema for surfaces



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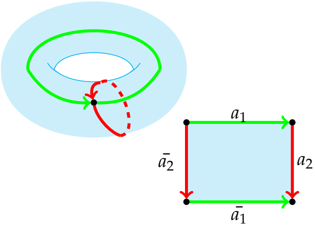

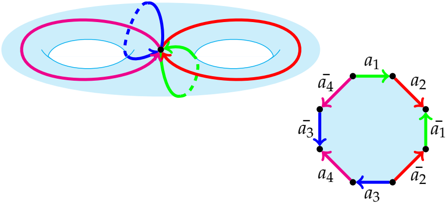

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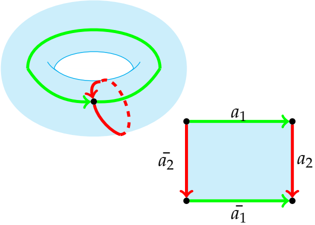
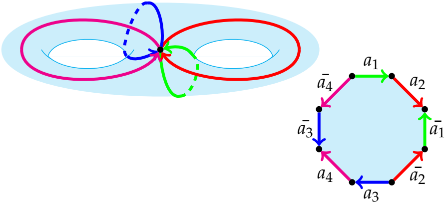


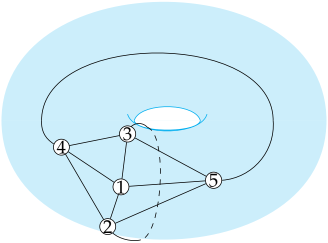


Classification Theorem

Every compact, connected surface of Euler genus $g \geqslant 1$ is homeomorphic to a polygonal surface given by one of the following **canonical signatures** σ:

- **Orientable:** $a_1 a_2 \bar{a}_1 \bar{a}_2 \dots a_{g-1} a_g \bar{a}_{g-1} \bar{a}_g$
- **Non-orientable:** $a_1 a_1 \dots a_g a_g$

Polygonal embedding for graphs



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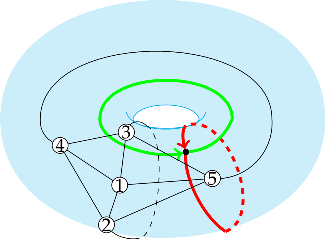

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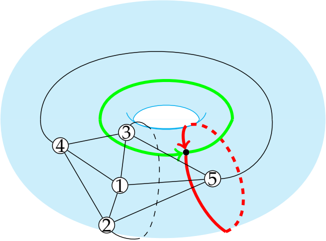

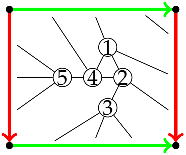

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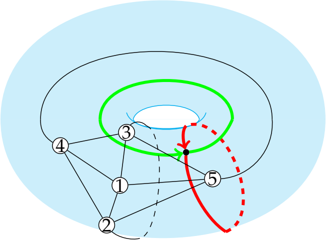

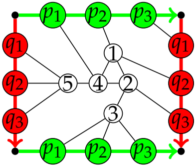

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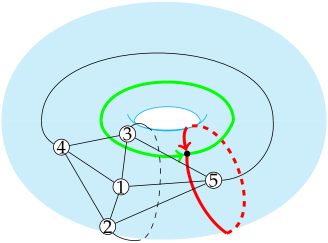

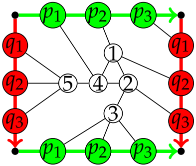

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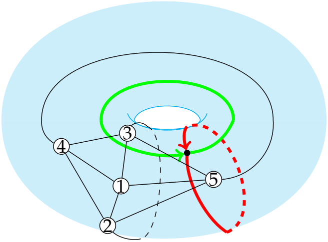

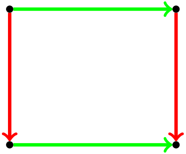

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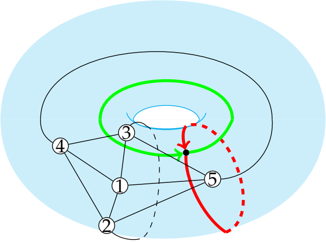

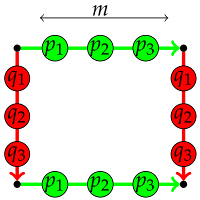

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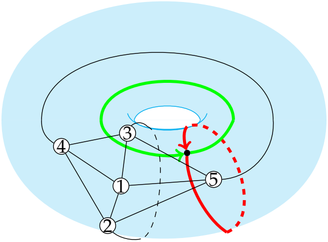

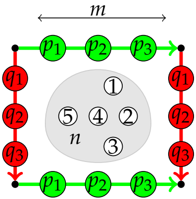

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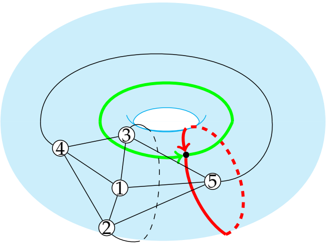

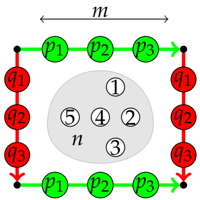

G has a **polygonal embedding** of type $P_\sigma(m,n)$:

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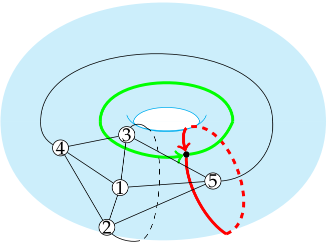

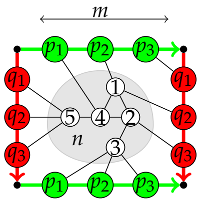

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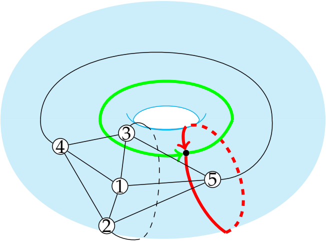

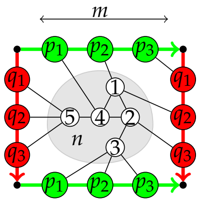

depends only on g and orientability

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$O(n+g)$ [LPVV01,FHdM22]

Sketch of the proof

Gavoille and H. (2023+)

For every n and every surface Σ of Euler genus $g \geqslant 1$, there is a graph embedded on Σ with $O(g^2(n+g)^2)$ vertices minor-universal for the n-graphs embeddable on Σ.

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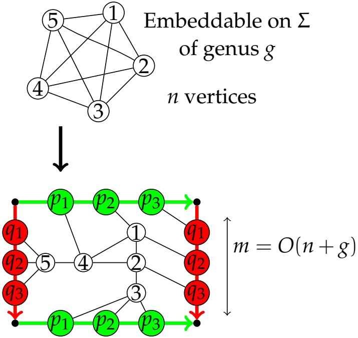

$P_\sigma(m,n)$ with $|\sigma| = 2g$

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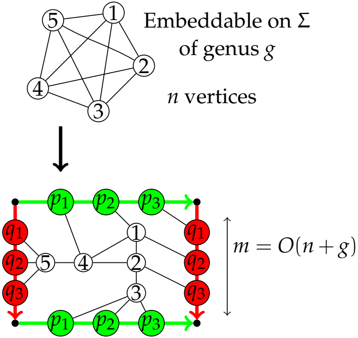
Technical theorem.

$\forall \sigma, m, n$, there is a graph with a polygonal embedding $P_\sigma(m+2n, |\sigma|^2(m+2n)^2)$, minor-universal for the graphs with a polygonal embedding $P_\sigma(m,n)$.



$P_\sigma(m,n)$ with $|\sigma| = 2g$

Step 1/2: getting an outerplanar-like embedding

Let G be a graph with a polygonal embedding $P_\sigma(m, n)$.
G is minor of a graph with polygonal embedding $P_\sigma(m+2n, 0)$.

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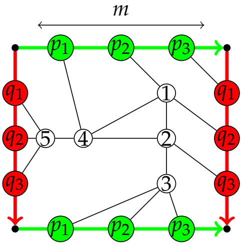

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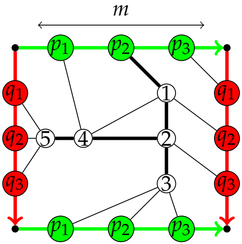

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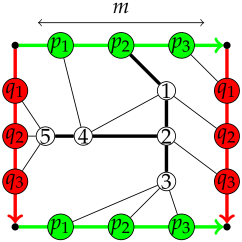
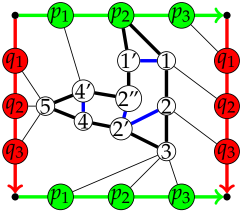


$P_\sigma(m,n)$



$P_\sigma(m,2n-1)$

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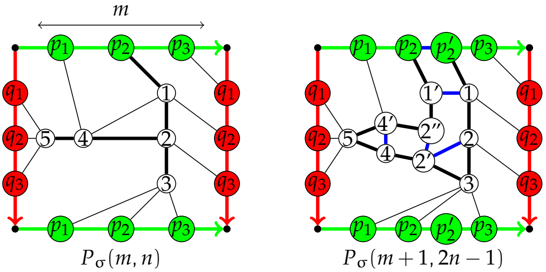

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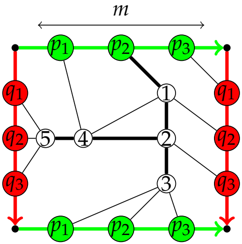
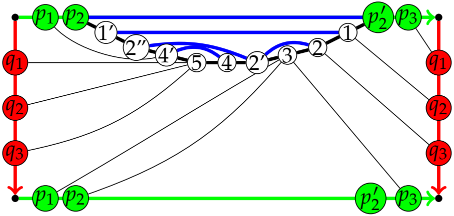


$P_\sigma(m,n)$



$P_\sigma(m+1,2n-1)$

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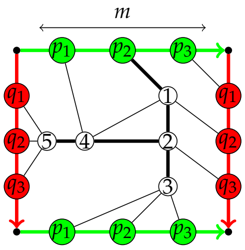
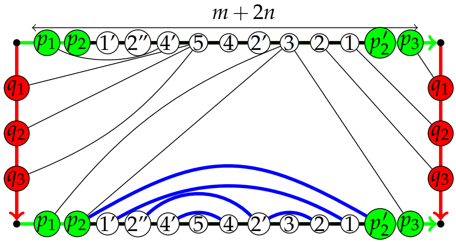
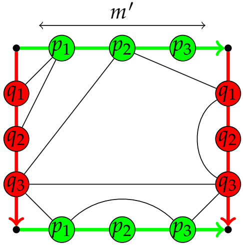
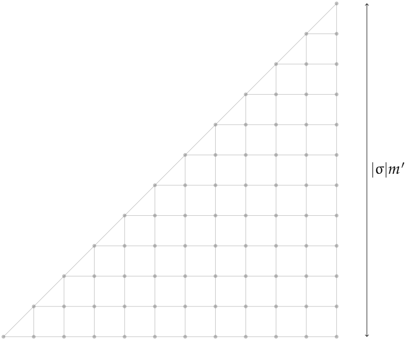




step 2/2: grid-like minor-universal graph

There is a graph with polygonal embedding $P_\sigma(m', \frac{(|\sigma|m')^2}{2})$ minor-universal for the graphs with polygonal embedding $P_\sigma(m', 0)$.





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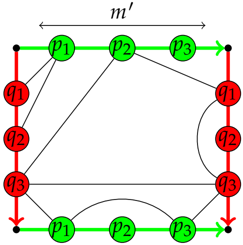

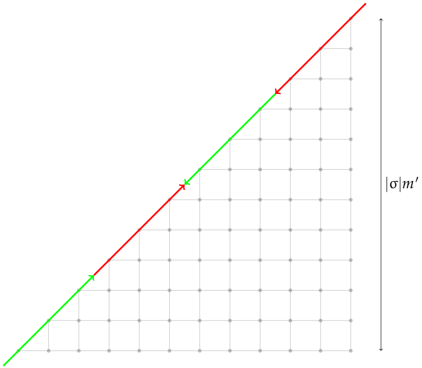

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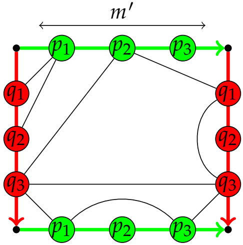

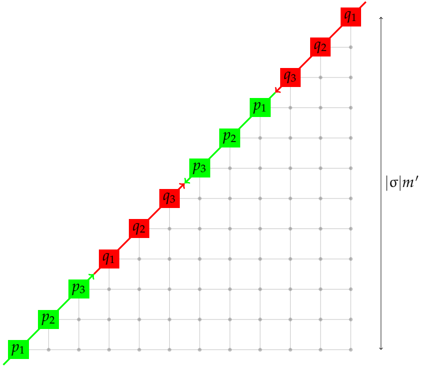

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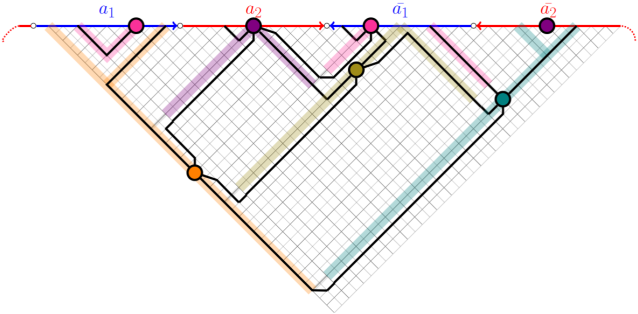

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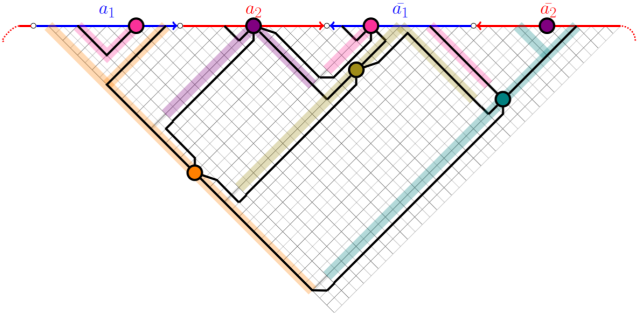

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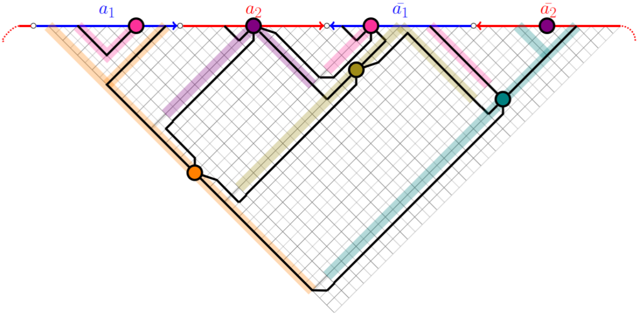

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- Extension to H-minor-free graphs?

Other contributions:

- Sparse kC_3-induced-minor-free graphs have logarithmic treewidth *(with M. Bonamy, E. Bonnet, H. Déprés, L. Esperet, C. Geniet, S. Thomassé, and A. Wesolek, in SODA'23)*
- Long induced paths in minor-closed graph classes and beyond *(with J.-F. Raymond, in Elec. J. of Comb. 2023)*
- On tree decompositions whose trees are minors *(with P. Blanco, L. Cook, M. Hatzel, F. Illingworth, and R. McCarty, arXiv)*
- On the proper interval completion problem within some chordal subclasses *(with F. Dross, I. Koch, V. Leoni, N. Pardal, M. I. L. Pujato, and V. F. dos Santos, arXiv)*

Thank you!